ANSWER TO HOMEWORK I

Solution 1. (1)Let

$$\frac{dx}{dt} = 5,$$

then along the characteristic curve $x(t) = 5t + a$,the partial differential equation becomes

$$\frac{du}{dt} = \frac{\partial u}{\partial t} + \frac{\partial u}{\partial x}\frac{dx}{dt} = e^{3t},$$

so that

$$u(x(t), t) = \frac{1}{3}e^{3t} + K,$$

where K is a constant, and $K = u(x(0), 0) - \frac{1}{3}$, so that

$$u(x(t), t) = \frac{1}{3}e^{3t} + u(x(0), 0) - \frac{1}{3} = \frac{1}{3}e^{3t} + e^{-a^2} - \frac{1}{3}.$$

Given the point (x, t), let $x = 5t + a$ be the unique characteristic curve passing through this point, then $a = x - 5t$ and the solution is

$$u(x, t) = \frac{1}{3}e^{3t} + e^{-(x-5t)^2} - \frac{1}{3}.$$

(2)Let

$$\frac{dx}{dt} = -x,$$

then along the characteristic curve $x(t) = x_0e^{-t}$,the partial differential equation becomes

$$\frac{du}{dt} = \frac{\partial u}{\partial t} + \frac{\partial u}{\partial x}\frac{dx}{dt} = 0,$$

so that

$$u(x(t), t) = K,$$

where K is a constant, and $K = u(x(0), 0) - \frac{1}{3}$, so that

$$u(x(t), t) = u(x(0), 0) = x_0^3 - 1.$$

Given the point (x, t), let $x = x_0e^{-t}$ be the unique characteristic curve passing through this point, then $x_0 = xe^t$ and the solution is

$$u(x, t) = x^3e^{3t} - 1.$$

Solution 2. (1)Since $5^2 - 4 \times 1 \times 6 = 1 > 0$,then the equation is of hyperbolic type. The characteristic equation is

$$\left(\frac{dy}{dx}\right)^2 - 5\frac{dy}{dx} + 6 = 0,$$

therefore let

$$\xi = 3x - y, \quad \eta = 2x - y,$$

then

$$\begin{aligned} u_x &= 3u_\xi + 2u_\eta, \\ u_y &= -u_\xi - u_\eta, \\ u_{xx} &= 9u_{\xi\xi} + 12u_{\xi\eta} + 4u_{\eta\eta}, \\ u_{xy} &= -3u_{\xi\xi} - 5u_{\xi\eta} - 2u_{\eta\eta}, \\ u_{yy} &= u_{\xi\xi} + 2u_{\xi\eta} + u_{\eta\eta}, \end{aligned}$$

substitute the above relations into the equation, we have

$$u_{\xi\eta} = 0,$$

which implies for any smooth functions F and G,

$$u = F(3x - y) + G(2x - y),$$

is a solution to the equation.

(2)Since $(2y)^2 - 4y^2 = 0$,then the equation is of hyperbolic type. The characteristic equation is

$$y^2 \left(\frac{dy}{dx}\right)^2 + 2y\frac{dy}{dx} + 1 = 0,$$

therefore let

$$\xi = \frac{y^2}{2} + x, \quad \eta = y,$$

then

$$\begin{aligned} u_x &= u_\xi, \\ u_y &= yu_\xi + u_\eta, \\ u_{xx} &= u_{\xi\xi}, \\ u_{xy} &= yu_{\xi\xi} + u_{\xi\eta}, \\ u_{yy} &= u_\xi + y^2 u_{\xi\xi} + 2yu_{\xi\eta} + u_{\eta\eta}, \end{aligned}$$

substitute the above relations into the equation, we have

$$u_{\xi\eta} = 6\eta,$$

which implies for any smooth functions F and G,

$$u = y^3 + yF\left(\frac{y^2}{2} + x\right) + G\left(\frac{y^2}{2} + x\right),$$

is a solution to the equation.

Solution 3. (1)By direct computation, we have

$$u_{xx} + u_{yy} = -\frac{2(x^2 - y^2)}{(x^2 + y^2)^2} + \frac{2(x^2 - y^2)}{(x^2 + y^2)^2} = 0,$$

which implies $u = \log(x^2 + y^2)$ is a harmonic function in $\mathbb{R}^2 \backslash \{(0, 0)\}$.

(2)Let $x = r\cos\theta$, $y = r\sin\theta$, then

$$\begin{aligned} u_x &= u_r \cos\theta - u_\theta \frac{\sin\theta}{r}, \\ u_y &= u_r \sin\theta + u_\theta \frac{\cos\theta}{r}, \end{aligned}$$

furthermore,

$$u_{xx} = u_{rr}(\cos\theta)^2 + u_r\frac{(\sin\theta)^2}{r} + u_\theta\frac{2\sin\theta\cos\theta}{r^2} - u_{r\theta}\frac{2\sin\theta\cos\theta}{r} + u_{\theta\theta}\frac{(\sin\theta)^2}{r^2},$$
$$u_{yy} = u_{rr}(\sin\theta)^2 + u_r\frac{(\cos\theta)^2}{r} - u_\theta\frac{2\sin\theta\cos\theta}{r^2} + u_{r\theta}\frac{2\sin\theta\cos\theta}{r} + u_{\theta\theta}\frac{(\cos\theta)^2}{r^2},$$

therefore

$$u_{xx} + u_{yy} = u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta},$$

which implies

$$\Delta = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}.$$

Therefore for $u = \log(x^2+y^2)$, we have

$$\Delta u = (\log r)_{rr} + \frac{1}{r}(\log r)_r = -\frac{2}{r^2} + \frac{2}{r^2} = 0,$$

which implies that $u = \log(x^2+y^2)$ is harmonic.

Solution 4. Suppose $u^2(x,y)$ attains its maximum $M > 0$ at $(x_0,y_0) \in D$. Let $v = u^2$, then

$$a(x,y)u_x(x,y) + b(x,y)u_y(x,y) = -u(x,y).$$

If $(x_0,y_0) \in \mathring{D}$, since $v(x,y)$ attains its maximum at (x_0,y_0), therefore

$$\nabla v(x_0,y_0) = 0,$$

which implies

$$v_x(x_0,y_0) = v_y(x_0,y_0) = 0,$$

therefore by the equation,

$$M = v(x_0,y_0) = 0,$$

which is a contradiction, therefore $M \le 0$, since $v = u^2 \ge 0$, therefore $u \equiv 0$.

If $(x_0,y_0) \in \partial D$, since $v(x,y)$ attains its maximum at (x_0,y_0) and $a(x_0,y_0)x_0 + b(x_0,y_0)y_0 > 0$, therefore let $\mathbf{l} = \left(a(x_0,y_0)^2 + b(x_0,y_0)^2\right)^{-\frac{1}{2}}(a(x_0,y_0), b(x_0,y_0))$, we have

$$\begin{aligned}\nabla_{\mathbf{l}} v(x_0,y_0) &= \left(a(x_0,y_0)^2 + b(x_0,y_0)^2\right)^{-\frac{1}{2}}[a(x_0,y_0)v_x(x_0,y_0) + b(x_0,y_0)v_y(x_0,y_0)] \\ &\ge 0,\end{aligned}$$

however, by the equation again

$$a(x_0,y_0)v_x(x_0,y_0) + b(x_0,y_0)v_y(x_0,y_0) = -v(x_0,y_0) = -M < 0,$$

which is a contradiction, therefore $M \le 0$,since $v = u^2 \ge 0$, therefore $u \equiv 0$.

Solution 5. (1)Since u is harmonic, the u_x and u_y are also harmonic, then by the mean value theorem,

$$\begin{aligned}|u_x(x_0,y_0)| &= \left|\frac{1}{|B(x_0,y_0)|}\iint_{B_r(x_0,y_0)} u_x dxdy\right| \\ &= \left|\frac{1}{|B(x_0,y_0)|}\int_{\partial B_r(x_0,y_0)} u\cdot n_x dS(x,y)\right| \\ &\le \frac{C}{r}\max_{\partial B_r(x_0,y_0)}|u|,\end{aligned}$$

where C is a sufficiently large constant independent of r. Similarly, we have

$$|u_y(x_0, y_0)| \le \frac{C}{r} \max_{\partial B_r(x_0,y_0)} |u|.$$

Therefore

$$|\nabla u(x_0, y_0)| \le \frac{C}{r} \max_{\partial B_r(x_0,y_0)} |u|.$$

(2)Since u is harmonic, then for arbitrary $(x, y) \in \mathbb{R}^2$,

$$|\nabla u(x, y)| \le \frac{C_1}{r} \max_{\partial B_r(x,y)} |u| \le \frac{C_2(1 + r^2)^{\frac{\gamma}{2}}}{r},$$

where C_1 and C_2 are two constants. Let r goes to infinity, we have

$$|\nabla u(x, y)| = 0,$$

therefore $u(x, y) \equiv Const.$